Q: ① How many 7-card hands have 2 hearts, 2 clubs, 2 spades, and one diamond?

Answer $\underbrace{\binom{13}{2}}_{\heartsuit}\ \underbrace{\binom{13}{2}}_{\clubsuit}\ \underbrace{\binom{13}{2}}_{\spadesuit}\ \underbrace{\binom{13}{1}}_{\diamondsuit}$ $= \binom{13}{1}$

$$\binom{13}{2} = \frac{13 \cdot 12}{2 \cdot 1} = 78$$

$$\boxed{(78)^3 \cdot 13}$$

Exercise ② Suppose $n, k \in \mathbb{N}$, $1 \le k \le n$.

Prove that $k\binom{n}{k} = n\binom{n-1}{k-1}$

ⓐ Using an algebraic proof

ⓑ Using a combinatorial proof.

ⓐ $LHS = k\binom{n}{k} = k\frac{n!}{k!(n-k)!} = \boxed{\frac{n!}{(k-1)!(n-k)!}}$

↳ $k!=k(k-1)(k-2)\cdots 2\cdot 1$

$RHS = n\binom{n-1}{k-1} = n \cdot \frac{(n-1)!}{(k-1)!\underbrace{((n-1)-(k-1))!}_{(n-k)!}} = \boxed{\frac{n!}{(k-1)!(n-k)!}}$

$= LHS$. ∎

ⓑ $LHS = \left(\begin{array}{c}\#\text{ of ways of choosing} \\ k \text{ elements from a} \\ \text{set of } n \text{ element}\end{array}\right) \& \left(\begin{array}{c}\text{Choosing} \\ \text{one} \\ \text{of the elements}\end{array}\right)$.

$\binom{n}{k}$ $\binom{k}{1} = k$

$= \#$ of ways to choose one element from n & then choose $k-1$ elements from those that are left.

$= \binom{n}{1} \cdot \binom{n-1}{k-1} = n\binom{n-1}{k-1}$. ✓

From Hint ③

Find # of positive integers $\leq n$ and relatively prime to n if $n = p \cdot q^3$ for prime #s p, q.

Solution: # of numbers $\leq n$ that have common factors with n

- $q^3 = \frac{n}{p}$ numbers with p as a factor.
- pq^2 numbers with q as a factor $= \frac{n}{q}$
- $q^2 = \frac{n}{pq}$ numbers with p and q as factors

# of numbers $\leq n$ with common factors p & q.

$$q^3 + pq^2 - q^2$$

# of numbers $\leq n$ relatively prime to n

$$= \boxed{n - q^3 - pq^2 + q^2}$$

(Q4) Find the number of solutions to $x_1 + x_2 + x_3 = 11$, where

(a) Each x_j is a nonnegative integer.

(b) Each x_j is a positive integer.

(x_1, x_2, x_3)

(13) (a) 13 slots → can put a * or | in each slot.
(but only 2 bars)

$$\binom{13}{2} = \binom{13}{2} = \frac{13 \cdot 12}{2 \cdot 1} = \boxed{78}$$

(b) Each x_j is positive — need each # to be at least 1.

8 stars left 2 bars

$$\binom{10}{2} = \frac{10 \cdot 9}{2 \cdot 1} = \boxed{45}.$$

(b) another way

(a) — # ways to get bar at beginning.
— # ways to get bar at end + * bar at both ends
— # ways to get 2 bars together.

$$= 78 - \binom{12}{1} - \binom{12}{1} + 1 - 11$$

$$= 78 - 12 - 12 + 1 - 11 = 45.$$

78
24
54

58
-11

Q5 There are 10 questions on the next DM2 exam. If the total score is 100 and each question is at least 5 points, how possible ways are there to assign the points on the questions?